

# THE REEB FOLIATION ARISES AS A FAMILY OF LEGENDRIAN SUBMANIFOLDS AT THE END OF A DEFORMATION OF THE STANDARD $S^3$ IN $S^5$

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**ABSTRACT.** We realize the Reeb foliation of  $S^3$  as a family of Legendrian submanifolds of the unit  $S^5 \subset \mathbb{C}^3$ . Moreover we construct a deformation of the standard contact  $S^3$  in  $S^5$ , via a family of contact submanifolds, into this realization.

## 1. INTRODUCTION

The Reeb foliation is a codimension one smooth foliation of the 3-sphere  $S^3$  obtained by glueing two Reeb components  $S^1 \times D^2$  and  $D^2 \times S^1$ . Since the one-sided holonomies of the Reeb components along  $\{1\} \times \partial D^2$  and  $\partial D^2 \times \{1\}$  are trivial, the Reeb foliation is not analytic (“Haefliger’s remark”).

On the other hand the 1-jet space  $J^1(\mathbb{R}^n, \mathbb{R}) \approx \mathbb{R}^{2n+1}$  for a function of  $n$  variables carries the canonical contact structure. It is contactomorphic to the unit sphere  $S^{2n+1} \subset \mathbb{C}^{n+1}$  minus any point. Here  $S^{2n+1}$  has the standard contact form  $\alpha = \sum_{i=1}^{n+1} r_i^2 d\theta_i | S^{2n+1}$  ( $r_i = |z_i|$ ,  $\theta_i = \arg z_i$  for coordinates  $z_i$  of  $\mathbb{C}^{n+1}$ ). Thus we may regard a codimension- $n$  submanifold  $M^{n+1} \subset S^{2n+1}$  as a system of  $n$  first-order partial differential equations (for implicit functions). If  $\alpha|_{M^{n+1}} = 0$  and  $\alpha|_{M^{n+1}} \neq 0$ , the system is completely integrable and regular, and therefore defines a codimension one foliation  $\mathcal{F}$  on  $M^{n+1}$ . The leaves of  $\mathcal{F}$  are Legendrian submanifolds of  $S^{2n+1}$  corresponding to the solutions.

In this article we construct an embedding of  $S^3$  into the standard  $S^5$  so that the image has the Reeb foliation  $\mathcal{F}$  by Legendrian submanifolds. This example shows that even a non-taut foliation can be a family of Legendrian submanifolds of  $J^1(\mathbb{R}^n, \mathbb{R})$ . Moreover we prove

**Theorem 1.1.** *There exists a smooth family  $\{M_t^3\}_{t \in [0, 3/2]}$  of codimension-2 submanifolds of  $S^5$  such that*

- (1)  $M_0^3$  is the standard  $S^3 (\subset \mathbb{C}^2 \subset \mathbb{C}^3)$ ,
- (2)  $M_t^3$  is an embedded contact submanifold for  $0 \leq t < 1$ ,
- (3)  $M_1^3$  admits a Reeb foliation by injectively immersed Legendrian submanifolds of  $S^5$ , and
- (4)  $-M_t^3$  is an embedded overtwisted contact submanifold for  $1 < t < 3/2$ .

The foliated submanifold  $M_1^3$  is obtained by joining two great circles  $\{r_1 = 1\}$ ,  $\{r_2 = 1\} \subset S^5$  through the Legendrian torus  $T = \{r_1 = r_2 = r_3 = 1/\sqrt{3}, \theta_1 + \theta_2 + \theta_3 = 0\}$ . The family  $M_t^3$  is obtained as a byproduct in the process of isotoping  $M_1 \subset S^5$  to the unknot. The author is seeking the converse approach, i.e., to find a foliated submanifold by using contact topology or open-books (see Remark 1 in §2).

## 2. PROOF AND REMARK

*Proof.* Let  $\pi$  be the natural projection of  $S^5$  to the 2-simplex  $\Delta = \{(r_1^2, r_2^2, r_3^2) | r_1^2 + r_2^2 + r_3^2 = 1\} \subset \mathbb{R}^3$ , which sends the Legendrian 2-torus  $T = \{r_1 = r_2 = r_3 = 1/\sqrt{3}, \theta_1 + \theta_2 + \theta_3 = 0\} \subset S^5$  to the barycenter  $G$ . The set  $\Gamma = \pi^{-1}(\partial\Delta)$  contains the great circles  $\pi^{-1}(\{V_1, V_2, V_3\})$  where  $V_i$  denotes the vertex  $r_i^2 = 1$ . Except them  $\pi|_{\Gamma}$  is a  $T^2$ -fibration. On the other hand,  $\pi|(S^5 \setminus \Gamma)$  is a

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$T^3$ -fibration. Now we take the coordinates  $(x, y)$  on  $\Delta$  by putting  $\overrightarrow{OP} = \overrightarrow{OG} + x\overrightarrow{GV_1} + y\overrightarrow{GV_2}$  for  $P \in \Delta$ , i.e.,

$$3r_1^2 = 1 + 2x - y (\geq 0), \quad 3r_2^2 = 1 - x + 2y (\geq 0), \quad \text{and} \quad 3r_3^2 = 1 - x - y (\geq 0).$$

Let  $M_0^3$  be the standard  $S^3 = \pi^{-1}(\overline{V_1 V_2})$ . We deform  $M_0^3$  with the help of a certain family of simple curves  $C_t : x = x_t(s), y = y_t(s), -\delta \leq s \leq \delta$  depicted in Fig.1 ( $0 < \delta \ll 1, 0 \leq t \leq 3/2$ ). Note that  $C_1$  has a break point  $G$  while  $x_1(s)$  and  $y_1(s)$  are smooth on  $(-\delta, \delta)$ .

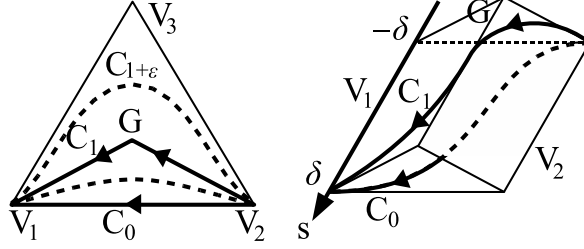


FIGURE 1. The curve  $C_t$  on  $\Delta$  and its parametrization by  $s$

We generate  $M_t^3 \subset S^5$  by moving the intersection of the “wall”  $W_s = \text{cl}\{\theta_1 + \theta_2 + \theta_3 = s\} \subset S^5$  with the fibre  $\pi^{-1}(x_t(s), y_t(s))$  for  $-\delta \leq s \leq \delta$ . Then we can see that  $M_t^3$  realizes the join of two large circles  $\pi^{-1}(V_2)$  and  $\pi^{-1}(V_1)$ . Now we give a precise definition of the curve  $C_t$ . Put  $\varphi_0(u) = \frac{1}{2}(1 + u)$  for  $u \in [-1, 1]$ , and take a smooth function  $\varphi_1(u)$  and a smooth odd function  $s(u)$  such that

$$\begin{aligned} \varphi_1(u) &= 0 \quad (-1 \leq u \leq 0), \quad \varphi_1'(u) > 0 \quad (0 < u \leq 1), \quad \varphi_1(u) = \varphi_0(u) \quad (1/2 \leq u \leq 1), \\ s'(u) &> 0 \quad (-1 < u < 1), \quad s(1) = \delta, \quad s(-1) = -\delta, \quad \text{and} \quad s(u) \text{ is } C^\infty\text{-tangent to } \pm\delta. \end{aligned}$$

The inverse function  $u(s)$  of  $s(u)$  is defined on  $[-\delta, \delta]$ . It is smooth on  $(-\delta, \delta)$  ( $u'(\pm\delta) = +\infty$ ). We put  $\varphi_t(u) = (1 - t)\varphi_0(u) + t\varphi_1(u)$ , and take the curve

$$C_t : \quad x = x_t(s) = \varphi_t(u(s)), \quad y = y_t(s) = \varphi_t(u(-s)), \quad -\delta \leq s \leq \delta.$$

Next we show that  $M_t^3$  is a smooth submanifold. By moving the 2-torus  $(M_t^3 \setminus \Gamma) \cap W_s$  for  $-\delta < s < \delta$ , we see that  $M_t^3 \setminus \Gamma$  is diffeomorphic to  $T^2 \times (-\delta, \delta)$ . Moreover  $M_t^3$  is topologically the join  $S^1 \star S^1 \approx S^3$ . Thus it only remains for us to examine the smoothness of  $M_t^3$  along  $M_t^3 \cap \Gamma$ . We restrict ourself to the connected component of  $M_t^3 \cap \Gamma$  corresponding to  $s = +\delta$  and omit the other component. We put

$$\widetilde{M}_t^3 : \begin{cases} r_1^2 + r_2^2 + r_3^2 = 1 \\ 3r_2^2 = 1 - \frac{1}{2}(1 + u) + (1 - t)(1 - u) \\ 3r_3^2 = 1 - \frac{1}{2}(1 + u) - \frac{1 - t}{2}(1 - u) = \frac{t}{3 - 2t} \cdot 3r_2^2 \\ \theta_1 + \theta_2 + \theta_3 = 1 \end{cases}$$

where  $u \in [1/2, 1]$  is a parameter to be eliminated. Then  $\{\theta_1 = \text{const}\} \subset \widetilde{M}_t^3$  is a smooth disk since it tangents to the real 2-plane  $\left\{ z_1 = \exp \sqrt{-1}\theta_1, \quad z_3 = \overline{z_2} \cdot \sqrt{\frac{t}{3 - 2t}} \exp \{\sqrt{-1}(1 - \theta_1)\} \right\} \subset \mathbb{C}^3$  at  $u = 1$ . Since the function  $s(u)$  smoothly tangents to  $\delta$  at  $u = 1$ ,  $M_t^3$  is a smooth 3-sphere.

Next we consider the (non-)integrability of the restriction  $\lambda_t = \alpha|_{M_t^3}$  of the standard contact form  $\alpha = r_1^2 d\theta_1 + r_2^2 d\theta_2 + r_3^2 d\theta_3|_{S^5}$ . Using  $(\theta_1, \theta_2, s)$  as coordinates of  $M_t^3 \setminus \Gamma$ , we can write

$$\lambda_t = x_t(s)d\theta_1 + y_t(s)d\theta_2 + (1 - x_t(s) - y_t(s))ds.$$

Here the sign of  $\lambda_t \wedge d\lambda_t$  with respect to  $d\theta_1 \wedge d\theta_2 \wedge ds > 0$  coincides with that of  $x_t'(s)y_t(s) - x_t(s)y_t'(s)$ , and that of  $1 - t$ . More generally, if a submanifold  $M^3 (\approx T^2 \times \mathbb{R}) \subset S^5$  is presented by a simple curve  $C : x = x(s), y = y(s)$  on  $\text{int}\Delta$ , the negative areal verocity  $x'(s)y(s) - x(s)y'(s)$

still presents the non-integrability of  $\alpha|M^3$ . In the case where  $t = 1$ , the integrability means the vanishing of the areal velocity. That is why the curve  $C_1$  is broken into two rays to/from the origin  $G$ , and  $M_1^3$  is non-analytic.

On the other hand, for cylindrical coordinates  $(\theta_1, (r_2, \theta_2))$ ,  $\mu_t = \alpha|\widetilde{M_t^3}$  and  $\mu_t \wedge d\mu_t$  are written as

$$\mu_t = \left(1 - \frac{3}{3-2t}r_2^2\right)d\theta_1 + \frac{3(1-t)}{3-2t}r_2^2d\theta_2 \quad \text{and} \quad \mu_t \wedge d\mu_t = \frac{6(1-t)}{3-2t}d\theta_1 \wedge (r_2dr_2 \wedge d\theta_2).$$

This implies that the sign of  $\lambda_t \wedge d\lambda_t$  everywhere coincides with that of  $1 - t$ .

Now we show that the foliation of  $M_1^3$  is a Reeb foliation. The definition of  $M_1^3$  is

$$\begin{cases} 3r_1^2 = 1 + 2\varphi(u(s)) - \varphi(u(-s)) \\ 3r_2^2 = 1 - \varphi(u(s)) + 2\varphi(u(-s)) \\ 3r_3^2 = 1 - \varphi(u(s)) - \varphi(u(-s)) \\ \theta_1 + \theta_2 + \theta_3 = s \end{cases}$$

where  $s \in [-\delta, \delta]$  is a parameter to be eliminated. On the open solid torus  $H = \{s > 0\} \subset M_1^3$ , we have

$$\alpha|H = \varphi(u(s))d\theta_1 + \{1 - \varphi(u(s))\}ds.$$

Thus the surface of  $\theta_2$ -revolution of the graph of  $\theta_1 = \int \frac{\varphi(u(s)) - 1}{\varphi(u(s))}ds$  is a leaf. Similarly, we can describe the foliation on  $\{s < 0\}$ . These foliations spiral into  $T$  and form a transversely oriented Reeb foliation, to which the positive Hopf link  $\{r_1 = 1\} \cup \{r_2 = 1\}$  is positively transverse.

Finally we see from  $d(\theta_1 + \theta_2) \wedge d\lambda_t = \{x'_t(s) - y'_t(s)\}d\theta_1 \wedge d\theta_2 \wedge ds > 0$  ( $t \neq 1$ ) that the positive Hopf band  $\ker(d\theta_1 + d\theta_2)$  is a supporting open-book for  $0 \leq t < 1$ . On the other hand, the negative Hopf band  $\ker(-d\theta_1 - d\theta_2)$  on  $-M_t(\approx S^3)$  is a supporting open-book for  $1 < t < 3/2$ . Thus  $-M_t^3$  is overtwisted. Indeed it has the half-Lutz tube  $\{x_t(s) \leq 0\}$ . Moreover, since we can reverse the orientation of  $S^3$  by a diffeotopy, we obtain the following “negative stabilization” lemma. This ends the proof.  $\square$

**Lemma 2.1.** *The overtwisted contact submanifold  $-M_{5/4}^3 \subset S^5$  is diffeotopic to the standard  $S^3 \subset S^5$ . Particularly  $-M_{5/4}^3$  is differential topologically unknotted, but contact topologically knotted.*

*Remark 1.* Any closed oriented 3-manifold admits an open-book decomposition (Alexander [1]). We can associate to it a contact structure (Thurston-Winkelnkemper [8]) as well as a spinnable foliation (see [5]). Further any contact structure is supported by an open-book decomposition (Giroux [3]). Using this result, the author constructed a certain immersion of any contact 3-manifold into  $J^1(\mathbb{R}^2, \mathbb{R})$  or  $S^5$  ([6]). This construction was generalized to any dimension, i.e.,  $M^{2n+1} \rightarrow J^1(\mathbb{R}^{2n}, \mathbb{R})$  or  $S^{4n+1}$  by Martínez Torres ([4]). The author proved that any/some contact structure of  $M^3$  can be deformed into some/any spinnable foliation ([5], see also [2]). He also proved that a certain higher dimensional contact structure can be deformed into a foliation ([7]). It is interesting to generalize the present result to these cases.

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